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# TRANSFORMATIONS APPROXIMATING A GROUP GENERATED BY THE LEVY LAPLACIAN(Quantum Stochastic Analysis and Related Fields)

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## TRANSFORMATIONS APPROXIMATING A GROUP GENERATED BY THE LÉVY LAPLACIAN

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### 1. Introduction

Since T. Hida [6] applied the Lévy Laplacian, which was introduced by P. Lévy [25], to his theory of generalized white noise functionals, this Laplacian has been studied within the framework of white noise calculus ([8,10,11,17,23,27,30,31], etc.). On the other hand, L. Accardi et al. [1] obtained a nice relation between the Laplacian and the Yang-Mills equation. It seems an interest to consider a relation to their results [1,2].

By H.-H. Kuo [16], an infinite dimensional Fourier-Mehler transform acting on the space  $(\mathcal{S})^*$  of generalized white noise functionals was introduced and he showed a relation between the transform and the Lévy Laplacian (see [19]). There are several Laplacian operators acting on  $(\mathcal{S})^*$ .

In this paper we discuss integral expressions of those Laplacians and groups generated by the Laplacians. In addition, we show a transform acting on  $(\mathcal{S})^*$  approximating a group generated by the Lévy Laplacian.

The paper is organized as follows. In Section 2 we assemble some basic notations of white noise calculus. In Section 3 we explain the definitions of Laplacian operators acting on Hida distributions, and give a limiting integral expression of the Lévy Laplacian with an integral expression of the Gross Laplacian. In Section 4 we define groups generated by the Laplacian operators acting on the Hida distributions and show that Kuo's Fourier-Mehler transform is given by a composition of groups generated by the number operator and the Gross Laplacian. In addition, we give a result that the group generated by the Lévy Laplacian is approximated by groups generated by the Gross Laplacian. Finally, in the last section we introduce a transform approximating a group generated by the Lévy Laplacian. This transform includes the adjoint operator of Kuo's Fourier-Mehler transform.

### 2. Preliminaries

In this section, we explain some basic notations of white noise analysis following [10,15,27,29]. We begin with a Gel'fand triple  $\mathcal{S} \subset L^2(\mathbf{R}) \subset \mathcal{S}^*$ , where  $\mathcal{S} \equiv \mathcal{S}(\mathbf{R})$  is the Schwartz space consisting of rapidly decreasing  $C^\infty$ -functions on  $\mathbf{R}$  and  $\mathcal{S}^* \equiv \mathcal{S}'(\mathbf{R})$  is its dual space. An operator  $A = -(d/du)^2 + u^2 + 1$  is a densely defined self-adjoint operator on  $L^2(\mathbf{R})$ . There exists an orthonormal basis  $\{e_\nu; \nu \geq 0\}$  for

$L^2(\mathbf{R})$  such that  $Ae_\nu = 2(\nu + 1)e_\nu$ . We define the norm  $|\cdot|_p$  by  $|f|_p = |A^p f|_0$  for  $f \in \mathcal{S}$  and  $p \in \mathbf{Z}$ , where  $|\cdot|_0$  is the  $L^2(\mathbf{R})$ -norm, and let  $\mathcal{S}_p$  be the completion of  $\mathcal{S}$  with respect to the norm  $|\cdot|_p$ . Then the dual space  $\mathcal{S}'_p$  of  $\mathcal{S}_p$  is the same as  $\mathcal{S}_{-p}$  (see [13]).

The Bochner-Minlos theorem admits the existence of a probability measure  $\mu$  on  $\mathcal{S}^*$  such that

$$C(\xi) \equiv \int_{\mathcal{S}^*} \exp\{i\langle x, \xi \rangle\} d\mu(x) = \exp\left\{-\frac{1}{2}|\xi|_0^2\right\}, \quad \xi \in \mathcal{S}.$$

The space  $(L^2) = L^2(\mathcal{S}^*, \mu)$  of complex-valued square-integrable functionals defined on  $\mathcal{S}^*$  admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where  $H_n$  is the space of multiple Wiener integrals of order  $n \in \mathbf{N}$  and  $H_0 = \mathbf{C}$ . This decomposition theorem says that each  $\varphi \in (L^2)$  is uniquely represented as

$$\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n), \quad f_n \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n},$$

where  $\mathbf{I}_n(f_n) \in H_n$  and  $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$  denotes the  $n$ -th symmetric tensor product of the complexification of  $L^2(\mathbf{R})$ .

For each  $p \in \mathbf{Z}$ , we define the norm  $\|\varphi\|_p$  of  $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n)$  by

$$\|\varphi\|_p^2 = \left( \sum_{n=0}^{\infty} n! |f_n|_{p,n}^2 \right)^{1/2}, \quad f_n \in \mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n}$$

where  $|\cdot|_{p,n}$  is the norm of  $\mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n}$  and  $\mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n}$  is the  $n$ -th symmetric tensor product of the complexification of  $\mathcal{S}_p$ . The norm  $\|\cdot\|_0$  is nothing but the  $(L^2)$ -norm. We put

$$(\mathcal{S}_p) = \{\varphi \in (L^2); \|\varphi\|_p < \infty\}$$

for  $p \in \mathbf{Z}, p \geq 0$ . Let  $(\mathcal{S}_p)^*$  be the dual space of  $(\mathcal{S}_p)$ . Then  $(\mathcal{S}_p)$  and  $(\mathcal{S}_p)^*$  are Hilbert spaces with the norm  $\|\cdot\|_p$  and the dual norm of  $\|\cdot\|_p$ , respectively. We define the space  $(\mathcal{S}_p)$  for  $p < 0$  by the completion of  $(L^2)$  with respect to  $\|\cdot\|_p$ . Then  $(\mathcal{S}_p), p < 0$ , is a Hilbert space with the norm  $\|\cdot\|_p$ . It is easy to see that for  $p > 0$ , the dual space  $(\mathcal{S}_p)^*$  of  $(\mathcal{S}_p)$  is given by  $(\mathcal{S}_{-p})$ . Moreover, we see that for any  $p \in \mathbf{R}$ ,

$$(\mathcal{S}_p) = \bigoplus H_n^{(p)},$$

where  $H_n^{(p)}$  is the completion of  $\{\mathbf{I}_n(f); f \in \mathcal{S}_{\mathbf{C}}^{\hat{\otimes} n}\}$  with respect to  $\|\cdot\|_p$ .

Denote the projective limit space of the  $(\mathcal{S}_p), p \in \mathbf{Z}, p \geq 0$ , and the inductive limit space of the  $(\mathcal{S}_p)^*, p \in \mathbf{Z}, p \geq 0$ , by  $(\mathcal{S})$  and  $(\mathcal{S})^*$ , respectively. Then  $(\mathcal{S})$  is

a nuclear space and  $(\mathcal{S})^*$  is nothing but the dual space of  $(\mathcal{S})$ . The space  $(\mathcal{S})^*$  is called the space of *Hida distributions* or *generalized white noise functionals*.

Since  $\exp \langle \cdot, \xi \rangle \in (\mathcal{S})$ , the  $S$ -transform is defined on  $(\mathcal{S})^*$  by

$$S[\Phi](\xi) = C(\xi) \ll \Phi, \exp \langle \cdot, \xi \rangle \gg, \xi \in \mathcal{S},$$

where  $\ll \cdot, \cdot \gg$  is the canonical pairing of  $(\mathcal{S})^*$  and  $(\mathcal{S})$ . In [10], we can see the following fundamental properties:

- i) if  $S[\Phi](\xi) = S[\Psi](\xi)$  for all  $\xi \in \mathcal{S}$ , then  $\Phi = \Psi$ .
- ii) if  $\Phi = \sum_{n=0}^{\infty} \Phi_n \in (\mathcal{S})^*$ , then there exist an integer  $p$  and distributions  $f_n \in \mathcal{S}_{C,p}^{\otimes n}$ ,  $n = 0, 1, 2, \dots$ , such that  $\sum_{n=0}^{\infty} n! |f_n|_{p,n}^2 < \infty$  and

$$S[\Phi](\xi) = \sum_{n=0}^{\infty} \langle \xi^{\otimes n}, f_n \rangle$$

for all  $\xi \in \mathcal{S}$ .

We denote the above Hida distribution  $\Phi_n$  in ii) by the same notation  $\mathbf{I}_n(f_n)$  for  $f_n \in \mathcal{S}_{C,p}^{\otimes n}$ .

### 3. Laplacian operators acting on Hida distributions

We introduce the definitions of Laplacian operators following [10] (see also [20]). Let  $F$  be a *Fréchet differentiable* function defined on  $\mathcal{S}$ , i.e. we assume that there exists a map  $F'$  from  $\mathcal{S}$  to  $\mathcal{S}^*$  such that

$$F(\xi + \eta) = F(\xi) + F'(\xi)(\eta) + o(\eta), \eta \in \mathcal{S},$$

where  $o(\eta)$  means that there exists  $p \in \mathbf{Z}$ ,  $p \geq 0$ , depending on  $\xi$  such that  $o(\eta)/|\eta|_p \rightarrow 0$  as  $|\eta|_p \rightarrow 0$ . If the first variation is expressed in the form

$$F'(\xi)(\eta) = \int_{\mathbf{R}} F'(\xi; u) \eta(u) du$$

for every  $\eta \in \mathcal{S}$  by using the generalized function  $F'(\xi; \cdot)$ , we define the *Hida derivative*  $\partial_t \Phi$  of  $\Phi$  to be the generalized white noise functional whose  $S$ -transform is given by  $F'(\xi; t)$ . The differentiation  $\partial_t$  is continuous from  $(\mathcal{S})$  into itself. Its adjoint operator  $\partial_t^*$  is continuous from  $(\mathcal{S})^*$  into itself.

Let  $(\mathcal{H}, \mathcal{B})$  be an abstract Wiener space. Suppose  $\psi$  is a real-valued twice  $\mathcal{H}$ -differentiable function on  $\mathcal{B}$  such that the second  $\mathcal{H}$ -derivative  $D^2\psi(x)$  at  $x$  is a trace class operator of  $\mathcal{H}$ . Then the *Gross Laplacian*  $\Delta_G$  ([4,5]) is defined by

$$\Delta_G \psi(x) = \text{Trace}_{\mathcal{H}} D^2 \psi(x).$$

The Laplacian  $\Delta_G$  has the expression  $\Delta_G \Phi = \int_{\mathbf{R}} \partial_t^2 \Phi dt$  on  $(\mathcal{S})$  (see [17]). The Gross Laplacian is a continuous linear operator from  $(\mathcal{S})$  into itself.

For any  $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})^*$ , the *number operator*  $N$  is defined by

$$N\Phi = \sum_{n=0}^{\infty} n\mathbf{I}_n(f_n).$$

The number operator is a continuous linear operator from  $(\mathcal{S})^*$  into itself. The operator  $N$  has the expression  $N\Phi = \int_{\mathbf{R}} \partial_t^* \partial_t \Phi dt$  on  $(\mathcal{S})$  (see [17]).

A Hida distribution  $\Phi$  is called an *L-functional* if for each  $\xi \in \mathcal{S}$ , there exist  $(S[\Phi])'(\xi; \cdot) \in L_{loc}^1(\mathbf{R}) \cap \mathcal{S}^*$ ,  $(S[\Phi])_s''(\xi; \cdot) \in L_{loc}^1(\mathbf{R}) \cap \mathcal{S}^*$  and  $(S[\Phi])_r''(\xi; \cdot, \cdot) \in L_{loc}^1(\mathbf{R}^2) \cap \mathcal{S}^*(\mathbf{R}^2)$  such that the first and second variations are uniquely expressed in the forms:

$$(S[\Phi])'(\xi)(\eta) = \int_{\mathbf{R}} (S[\Phi])'(\xi; u) \eta(u) du,$$

and

$$\begin{aligned} (S[\Phi])''(\xi)(\eta, \zeta) &= \int_{\mathbf{R}} (S[\Phi])_s''(\xi; u) \eta(u) \zeta(u) du \\ &+ \int_{\mathbf{R}^2} (S[\Phi])_r''(\xi; u, v) \eta(u) \zeta(v) dudv, \end{aligned}$$

for each  $\eta, \zeta \in \mathcal{S}$ , respectively and for any finite interval  $T$ ,  $\int_T (S[\Phi])_s''(\cdot; u) du$  is in  $S[(\mathcal{S})^*]$ . For any *L-functional*  $\Phi \in D_L$  and any finite interval  $T$  in  $\mathbf{R}$ , the *Lévy Laplacian*  $\Delta_L^T$  is defined by

$$\Delta_L^T \Phi = S^{-1} \left[ \frac{1}{|T|} \int_T (S[\Phi])_s''(\cdot; u) du \right].$$

This Laplacian has the following interesting properties.

- 1)  $\Delta_L^T = 0$  on  $(L^2)$  (see [7, 26]).
- 2)  $\Delta_L^T$  is a derivation under the Wick product (see [23]).

A Hida distribution  $\Phi$  is called to be *normal* if its *S-transform*  $S[\Phi]$  is given by a finite linear combination of

$$\int_{T^k} f(u_1, \dots, u_k) \xi(u_1)^{p_1} \dots \xi(u_k)^{p_k} du_1 \dots du_k, \quad (3.1)$$

where  $T$  is a finite interval in  $\mathbf{R}$ ,  $f \in L^1(T^k)$  and  $p_1, \dots, p_k \in \mathbf{N} \cup \{0\}$ ,  $k \in \mathbf{N}$ . For any  $p \geq 1$ , the normal functional with the *S-transform* given as in (3.1) is in  $D_L^T \cap (\mathcal{S}_{-p})$ , because the kernel

$$\int_{T^k} f(u_1, \dots, u_k) \delta_{u_1}^{\otimes p_1} \otimes \dots \otimes \delta_{u_k}^{\otimes p_k} du_1 \dots du_k$$

is in  $\mathcal{S}_{-1}^{\otimes(p_1+\dots+p_k)}$ . This functional plays the role of the polynomial in the infinite dimensional analysis. Let  $\mathcal{N}_T$  denote the set of all normal functionals in  $D_L^T$ . For  $p \geq 1$  and  $\Phi \in D_L^T$ , we define a  $(-p)$ -norm  $||| \cdot |||_{-p}$  by

$$||| \Phi |||_{-p}^2 = \sum_{k=0}^{\infty} \|(\Delta_L^T)^k \Phi\|_{-p}^2 \quad (\in [0, \infty])$$

and denote the completion of  $\mathcal{N}_T$  with respect to the norm  $||| \cdot |||_{-p}$  by  $\mathbf{D}_{-p}$ . Then  $\mathbf{D}_{-p}$  is the Hilbert space with the norm  $||| \cdot |||_{-p}$  and  $\Delta_L^T$  is a bounded linear operator from  $\mathbf{D}_{-p}$  into itself satisfying  $||| \Delta_L^T \Phi |||_{-p} \leq ||| \Phi |||_{-p}$  for  $\Phi \in \mathbf{D}_{-p}$ . We put  $\mathbf{D} = \bigcup_{p=1}^{\infty} \mathbf{D}_{-p}$  with the inductive limit topology. Then the Laplacian  $\Delta_L^T$  is a continuous linear operator on  $\mathbf{D}$ .

Let  $D_L^T$  denote the set of all  $L$ -functionals  $\Phi$  satisfying  $S[\Phi](\eta) = 0$  for  $\eta$  with  $\text{supp}(\eta) \subset T^c$ . In [22], Kuo obtained the following result.

**Theorem 3.1.** *Suppose  $\{j_\epsilon; \epsilon > 0\}$  is a family of continuous linear operators from  $\mathcal{S}^*$  into  $\mathcal{S}$  satisfying the following conditions:*

- (a)  $j_\epsilon^* \rightarrow I$  strongly on  $L^2(\mathbf{R})$  as  $\epsilon \rightarrow 0$ .
- (b)  $\lim_{\epsilon \rightarrow 0} |j_\epsilon|_{HS}^{-2} |j_\epsilon^* j_\epsilon|_{HS} = 0$ .
- (c) *There exists a uniformly bounded orthonormal basis  $\{e_k; k \geq 0\}$  for  $L^2(T)$  such that as  $\epsilon \rightarrow 0$ ,*

$$|j_\epsilon|_{HS}^{-2} \sum_{k=0}^{\infty} (j_\epsilon e_k)(t)^2 \longrightarrow \frac{1}{|T|} \text{ in } L^2(T).$$

Then for any  $\Phi$  in  $D_L^T$ ,

$$S[\Delta_L^T \Phi](\xi) = \lim_{\epsilon \rightarrow 0} |j_\epsilon|_{HS}^{-2} S[\Delta_G S^{-1}(S[\Phi] \circ j_\epsilon)](\xi).$$

If  $\varphi \in (\mathcal{S})$ , the functional  $S[\varphi]''(\xi)(\eta, \zeta)$ ,  $\eta, \zeta \in \mathcal{S}$  has an extension  $S[\varphi]''(\xi)(x, y)$ ,  $x, y \in \mathcal{S}^*$ , such that  $S[\varphi]''(\xi)(x, x)$  is in  $(\mathcal{S})$ .

The chaos expansions of  $\Delta_G \varphi$  and  $S[\varphi]''(\xi)(x, x)$  for  $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n)$  in  $(\mathcal{S})$  are given by

$$\Delta_G \varphi = \sum_{n=0}^{\infty} \mathbf{I}_n \left( (n+2)(n+1) \int_{\mathbf{R}} f_{n+2}(\cdot, t, t) dt \right)$$

and

$$S[\varphi]''(\xi)(x, x) = \sum_{n=0}^{\infty} n(n-1) \int_{\mathbf{R}^n} f_n(\mathbf{u}) \xi(u_1) \cdots \xi(u_{n-2}) x(u_{n-1}) x(u_n) d\mathbf{u},$$

respectively. Hence the expectation of  $S[\varphi]''(\xi)(\cdot, \cdot)$  is given by

$$\int_{\mathcal{S}^*} S[\varphi]''(\xi)(x, x) d\mu(x) = \sum_{n=0}^{\infty} n(n-1) \int_{\mathbf{R}^{n-1}} f_n(\mathbf{v}, t, t) \xi^{\otimes(n-2)}(\mathbf{v}) d\mathbf{v} dt.$$

Thus we come to get Lemma 3.2.

**Lemma 3.2.** *For any  $\varphi \in (\mathcal{S})$ , we have*

$$S[\Delta_G \varphi](\xi) = \int_{\mathcal{S}^*} S[\varphi]''(\xi)(x, x) d\mu(x).$$

We introduce an operator  $J_\epsilon$  on  $(\mathcal{S})^*$  into  $(\mathcal{S})$  by

$$S[J_\epsilon \Phi](\xi) = S[\Phi](j_\epsilon(\xi)), \quad \Phi \in (\mathcal{S})^*.$$

Using the operator  $J_\epsilon$ , we can obtain the following result.

**Theorem 3.3.** Let  $T$  be a finite interval in  $\mathbf{R}$  and  $\Phi$  an  $L$ -functional in  $D_L^T$ . Then we have

$$S[\Delta_L^T \Phi](\xi) = \lim_{\epsilon \rightarrow 0} (\theta_\epsilon)^2 \int_{S^*} S[J_\epsilon \Phi]''(\xi)(x, x) d\mu(x),$$

where  $\theta_\epsilon = |j_\epsilon|_{HS}^{-1}$ .

#### 4. Groups generated by infinite dimensional Laplacians

We now introduce an operator  $e^{z\Delta_G}$ ,  $z \in \mathbf{C}$  by

$$e^{z\Delta_G} \Phi = \sum_{n=0}^{\infty} \frac{(z\Delta_G)^n}{n!} \Phi$$

for  $\Phi \in (\mathcal{S})$ . This operator satisfies the following properties.

**Theorem 4.1 [32].** The  $e^{z\Delta_G}$  is a continuous linear operator from  $(\mathcal{S})$  into itself given by

$$e^{z\Delta_G} \Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(\ell_n(\Phi; z)), \quad \ell_n(\Phi; z) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} z^m T r^{\otimes m} * f_{n+2m} \quad (4.1)$$

for  $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})$ .

**Theorem 4.2 [32].** For any  $\Phi \in (\mathcal{S})$ , we have

$$S[e^{\frac{z}{2}\Delta_G} \Phi](\xi) = \int_{S^*} S[\Phi](\xi + \sqrt{z}x) d\mu(x),$$

where the integral is defined independent of choices of the branch of  $\sqrt{z}$  since  $\mu$  is symmetric.

An infinite dimensional Fourier-Mehler transform  $\mathbf{F}_\theta$ ,  $\theta \in \mathbf{R}$ , on  $(\mathcal{S})^*$  was defined by H.-H. Kuo [19] as follows. The transform  $\mathbf{F}_\theta \Phi$ ,  $\theta \in \mathbf{R}$  of  $\Phi \in (\mathcal{S})^*$  is defined by the unique Hida distribution with the  $S$ -transform

$$S[\mathbf{F}_\theta \Phi](\xi) = S[\Phi](e^{i\theta} \xi) \exp \left[ \frac{i}{2} e^{i\theta} \sin \theta |\xi|_0^2 \right], \quad \xi \in \mathcal{S}.$$

Moreover, the adjoint operator  $\mathbf{F}_\theta^*$  of  $\mathbf{F}_\theta$  is given by

$$\mathbf{F}_\theta^* \Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(h_n(\Phi; \theta)) \text{ for } \Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S}),$$

where

$$h_n(\Phi; \theta) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} \left( \frac{i}{2} \sin \theta \right)^m e^{i(m+n)\theta} T r^{\otimes m} * f_{n+2m};$$

$$Tr = \int_{\mathbf{R}} \delta_t \otimes \delta_t dt.$$

This operator  $\mathbf{F}_\theta^*$  is a continuous linear operator on  $(\mathcal{S})$ . (For details, see [19] and also [9].) The operator  $e^{i\theta N}$  is called the *Fourier-Wiener transform*, which is given by

$$e^{i\theta N} \Phi = \sum_{n=0}^{\infty} e^{in\theta} \Phi_n$$

for  $\Phi = \sum_{n=0}^{\infty} \Phi_n \in (\mathcal{S})$  (see [9]). The families  $\{e^{i\theta \Delta_G}; \theta \in \mathbf{R}\}$ ,  $\{e^{i\theta N}; \theta \in \mathbf{R}\}$  and  $\{\mathbf{F}_\theta^*; \theta \in \mathbf{R}\}$  are groups generated by  $i\Delta_G$ ,  $iN$  and  $iN + \frac{i}{2}\Delta_G$ , respectively (see [9]). Take  $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})$ . From (4.1), we see that

$$e^{\frac{i}{2}(e^{i\theta} \sin \theta) \Delta_G} \Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(\ell_n(\Phi; \frac{i}{2} e^{i\theta} \sin \theta)).$$

Hence,

$$e^{i\theta N} (e^{\frac{i}{2}(e^{i\theta} \sin \theta) \Delta_G} \Phi) = \sum_{n=0}^{\infty} \mathbf{I}_n(e^{in\theta} \ell_n(\Phi; \frac{i}{2} e^{i\theta} \sin \theta)).$$

Since  $e^{in\theta} \ell_n(\Phi; \frac{i}{2} e^{i\theta} \sin \theta) = h_n(\Phi; \theta)$ , we obtain the following relation.

**Theorem 4.3** [31].

$$\mathbf{F}_\theta^* = e^{i\theta N} \circ e^{\frac{i}{2}(e^{i\theta} \sin \theta) \Delta_G}.$$

**Remark:** Details of Lie algebras containing  $\Delta_G$  and  $N$  are discussed in [28].

A  $(C_0)$ -group  $\{G_t, t \in \mathbf{R}\}$  is given by

$$G_t = \lim_{\epsilon \rightarrow 0} \sum_{k=0}^n \frac{t^k}{k!} (\Delta_L^T)^k,$$

as an operator on  $\mathbf{D}$ . The group  $G_t$  has naturally an analytic extension  $G_z$ ,  $z \in \mathbf{C}$ . It is easily checked that for any  $\Phi \in \mathbf{D}$  and  $t \in \mathbf{R}$  there exists  $p \geq 1$  such that  $|||G_z \Phi|||_{-p} \leq e^{|z|} |||\Phi|||_{-p}$ .

An characterization of Hida distributions was obtained by J. Potthoff and L. Streit [29]. They say that for any  $F$  in  $S[(\mathcal{S})^*]$  and  $\xi, \eta$  in  $\mathcal{S}$ , the function  $F(\xi + \lambda\eta)$ ,  $\lambda \in \mathbf{R}$ , extends to an entire function  $F(\xi + z\eta)$ ,  $z \in \mathbf{C}$ . We define an operator  $g_z$ ,  $z \in \mathbf{C}$ , acting on a Hida distribution  $\Phi$  by

$$S[g_z \Phi](\xi) = \lim_{\epsilon \rightarrow 0} S[e^{z(\theta_\epsilon)^2 \Delta_G} J_\epsilon \Phi](\xi)$$

if the limit exists in  $S[(\mathcal{S})^*]$ . For  $\Phi \in \mathcal{N}_T$  and  $z \in \mathbf{C}$ , we have  $g_z \Phi \in \mathcal{N}_T$ . For  $p \geq 1$ , let  $\mathcal{E}_{-p}$  denote the collection of Hida distributions  $\Phi = \sum_{n=0}^{\infty} \Phi_n$  in  $(\mathcal{S}_{-p})$  such that  $\Phi_n \in \mathcal{N}_T \cap H_n^{(-p)}$ ,  $n = 0, 1, 2, \dots$ , and  $\sum_{n=0}^{\infty} |||\Phi_n|||_{-p} < \infty$ . Set  $\mathcal{E} = \bigcup_p \mathcal{E}_{-p}$ . It is clear that  $\mathcal{E}_{-p} \subset \mathbf{D}_{-p}$  for  $p \geq 1$  and  $\mathcal{E} \subset \mathbf{D}$ . By calculations of  $g_z \Phi$  and  $G_z \Phi$  for  $\Phi$  whose  $S$ -transform  $S\Phi$  is given as in (3.1), we get  $g_z = G_z$  on  $\mathcal{N}_T$  for  $z \in \mathbf{C}$ . The continuity of  $G_z$  implies the following result.



**Theorem 4.4.** *If  $\Phi = \sum_{n=0}^{\infty} \Phi_n$  is in  $\mathcal{E}_{-p}$  for  $p \geq 1$ , then  $\sum_{n=0}^{\infty} g_z \Phi_n \in \mathbf{D}_{-p}$  and  $G_z \Phi = \sum_{n=0}^{\infty} g_z \Phi_n$  for  $z \in \mathbf{C}$ . Moreover if*

$$\sum_{n=0}^{\infty} \sup_{\epsilon} \int_{\mathcal{S}^*} |S[J_{\epsilon} \Phi_n](\xi + \sqrt{2z} \theta_{\epsilon} x)| d\mu(x) < \infty$$

*holds for any  $z \in \mathbf{C}$  and  $\xi \in \mathcal{S}$ , then  $g_z \Phi$  exists in  $\mathbf{D}_{-p}$  and  $g_z \Phi = G_z \Phi$ .*

## 5. A generalization

For any  $\varphi \in (\mathcal{S})$ ,  $\xi \in \mathcal{S}$  and  $z_1, z_2 \in \mathbf{C}$ , the functional  $S[\varphi](z_1 \xi + z_2 \eta)$ ,  $\eta \in \mathcal{S}$ , can be extended to a functional  $\widetilde{S[\varphi]}(z_1 \xi + z_2 y)$ ,  $y \in \mathcal{S}^*$ , in  $(\mathcal{S})$  (cf. [15]). We denote this functional by the same symbol  $S[\varphi](z_1 \xi + z_2 y)$ . Thus we can define an operator  $\mathcal{G}_{\alpha, \beta}$  from  $(\mathcal{S})$  into itself by

$$S[\mathcal{G}_{\alpha, \beta} \varphi](\xi) = \int_{\mathcal{S}^*} S[\varphi](\alpha \xi + \beta x) d\mu(x). \quad (5.1)$$

Here we note that the right hand side of (5.1) is in  $S[(\mathcal{S})]$ . If  $\alpha = 1$  or  $-1$ ,  $\mathcal{G}_{\alpha, \beta}$  is equal to Lee's transform  $\mathcal{L}_{\alpha, \beta}$  ([24]) given by

$$\mathcal{L}_{\alpha, \beta} \varphi(x) = \int_{\mathcal{S}^*} \varphi(\alpha x + \beta y) d\mu(y), \quad \varphi \in (\mathcal{S}).$$

The transform  $\mathcal{L}_{\alpha, \beta}$  is applied to the heat equation associated with the operator  $(a\Delta_G + bN)^k$ ,  $k \geq 1$ ,  $a, b \in \mathbf{C}$  with  $\operatorname{Re} b^k \leq 0$ . (For details, see [3] and [14].) By the proof analogous to that of Theorem 3.2 in [32], we can obtain the following Lemma.

**Lemma 5.1.** *If a Hida distribution  $\Phi$  is in  $\mathcal{N}_T$ , then*

$$\lim_{\epsilon \rightarrow 0} \int_{\mathcal{S}^*} S[J_{\epsilon} \Phi](\alpha_{\epsilon}(z) \xi + \beta_{\epsilon}(z) x) d\mu(x) = S[g_z \Phi](\xi)$$

*holds for any  $\xi \in \mathcal{S}$ , where  $\alpha_{\epsilon}(z)$  and  $\beta_{\epsilon}(z)$  are complex-valued functions of  $z \in \mathbf{C}$  depending  $\epsilon > 0$  such that  $\alpha_{\epsilon}(z) \rightarrow 1$  and  $\beta_{\epsilon}(z)/\theta_{\epsilon} \rightarrow \sqrt{2it}$  as  $\epsilon \rightarrow 0$ .*

*Proof.* The proof comes from Theorem 4.4 and the following formula:

$$\int_{\mathcal{S}^*} S[\varphi](\alpha \xi + \beta x) d\mu(x) = S[e^{N \log \alpha} \circ e^{\frac{\beta^2}{2} \Delta_G} \varphi](\xi), \quad \varphi \in (\mathcal{S}), \alpha, \beta \in \mathbf{C}.$$

□

By Lemma 5.1, we have the following result which is a generalization of Theorem 4.7 in [32].

**Theorem 5.2.** Let  $\Phi$  be a Hida distribution in  $\mathcal{E}$  satisfying the condition

$$\sum_{n=0}^{\infty} \sup_{\epsilon} \int_{S^*} |S[J_{\epsilon}\Phi_n](\alpha_{\epsilon}(z)\xi + \beta_{\epsilon}(z)x)| d\mu(x) < \infty.$$

Then

$$\lim_{\epsilon \rightarrow 0} S[\mathcal{G}_{\alpha_{\epsilon}(z), \beta_{\epsilon}(z)} J_{\epsilon}\Phi](\xi) = S[G_z\Phi](\xi), \quad z \in \mathbb{C}, \xi \in \mathcal{S}. \quad (5.2)$$

*Proof.* From the assumption and the Lebesgue convergence theorem, we can calculate as follows:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} S[\mathcal{G}_{\alpha_{\epsilon}(z), \beta_{\epsilon}(z)} J_{\epsilon}\Phi](\xi) &= \lim_{\epsilon \rightarrow 0} \int_{S^*} S[J_{\epsilon}\Phi](\alpha_{\epsilon}(z)\xi + \beta_{\epsilon}(z)x) d\mu(x) \\ &= \sum_{n=0}^{\infty} \lim_{\epsilon \rightarrow 0} \int_{S^*} S[J_{\epsilon}\Phi_n](\alpha_{\epsilon}(z)\xi + \beta_{\epsilon}(z)x) d\mu(x). \end{aligned}$$

Consequently, by Lemma 5.1, we obtain (5.2).  $\square$

Theorem 4.3 admits an integral expression of the adjoint operator of Kuo's Fourier-Mehler transform:

$$S[\mathbf{F}_{\theta}^*\varphi](\xi) = \int_{S^*} S[\varphi](e^{i\theta}\xi + \sqrt{ie^{i\theta}\sin\theta}x) d\mu(x), \quad \varphi \in (\mathcal{S}).$$

Hence Theorem 5.2 implies the following

**Corollary 5.3.** Let  $\Phi$  be a Hida distribution in  $\mathcal{E}$  satisfying the condition in Theorem 5.2 with

$$\alpha_{\epsilon}(it) = e^{2it(\theta_{\epsilon})^2} \quad \text{and} \quad \beta_{\epsilon}(it) = \sqrt{ie^{2it(\theta_{\epsilon})^2} \sin(2t(\theta_{\epsilon})^2)}.$$

Then

$$\lim_{\epsilon \rightarrow 0} S[\mathbf{F}_{2t(\theta_{\epsilon})^2}^* J_{\epsilon}\Phi](\xi) = S[G_{it}\Phi](\xi), \quad t \in \mathbb{R}, \xi \in \mathcal{S}.$$

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